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SOLUTIONS OF THE SAINT-VENANT PROBLEM FOR A CYLINDER WITH HELICAL ANISOTROPY†

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Solutions of the Saint-Venant problem for a cylinder with helical anisotropy are presented in the form of a linear combination of elementary homogeneous solutions, the construction of which is reduced to boundary-value problems for ordinary differential equations with variable coefficients. These problems are integrated using analytical and numerical methods, and the elements of the stiffness matrix are investigated over a wide range of parameters. It is established that when the cylinder is stretched the sign and value of the torsional deformation depends considerably on the value of the relative angle of twist of the helices.

A cylinder with helical anisotropy can be represented, in particular, as the result of the helical winding of layers of thin filaments made of rigid material on a cylindrical surface, with a simultaneous coating of these with a polymer material. Homogenization methods [1, 2] can be used to determine the elastic characteristics. Hence, a transversely isotropic material is obtained with an axis of symmetry directed along the tangent to the helices. When a cylinder with helical anisotropy is stretched or compressed, in addition to longitudinal deformation there will also be torsional deformation and, conversely, under torsion in addition to torsional deformation there will also be longitudinal deformation. Naturally twisted rods will also possess similar properties [3]. Hence such rods can be used in devices which convert longitudinal deformations into longitudinally twisted deformations and vice versa.

1. FUNDAMENTAL RELATIONS OF THE THEORY OF ELASTICITY IN THE CASE OF HELICAL ANISOTROPY AND FORMULATION OF THE PROBLEM FOR A CYLINDER WITH HELICAL ANISOTROPY

Consider a cylindrical body, occupying a volume $V = S \times [0, L]$, where S is the transverse cross-section of the cylinder and L is its length. We will denote the side surface by $\Gamma = \partial S \times [0, L]$, where ∂S is the boundary of S. We connect the origin of a Cartesian system of coordinates x_1, x_2, x_3 with the geometrical centre of gravity of one of the ends of the cylinder. We will call this system of coordinates the fundamental system. To describe the helical anisotropy we will introduce an accompanying cylindrical system of coordinates r, θ, z, connected with the fundamental system by the relations

$$\begin{aligned} x_1 &= r\cos\theta\cos\tau z - r\sin\theta\sin\tau z \\ x_2 &= r\cos\theta\sin\tau z + r\sin\theta\cos\tau z \\ x_3 &= z \end{aligned} \tag{1.1}$$

Here and below we assume that $\tau = \text{const}$.

When $r = \text{const}$ and $\theta = \text{const}$ relations (1.1) are the parametric equations of a helical curve, where $\tau = 2\pi/h$, where h is the pitch of the helix. We will represent the radius vector of points on the helical curve in the form

$$\mathbf{R} = r\mathbf{e}_1' + z\mathbf{e}_3'$$

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Here

$$\mathbf{e}'_1 = \mathbf{e}_r = \mathbf{i}_1(\cos\theta\cos\tau z - \sin\theta\sin\tau z) + \mathbf{i}_2(\cos\theta\sin\tau z + \sin\theta\cos\tau z)$$
$$\mathbf{e}'_2 = \mathbf{e}_\theta = -\mathbf{i}_1(\sin\theta\cos\tau z + \cos\theta\sin\tau z) + \mathbf{i}_2(\cos\theta\cos\tau z - \sin\theta\cos\tau z)$$
$$\mathbf{e}'_3 = \mathbf{e}_z$$

and i_n are the unit vectors of the fundamental system of coordinates.

We will connect the following natural frame of reference with the helical curve

$$\mathbf{e}_1 = \mathbf{n}, \quad \mathbf{e}_2 = \mathbf{b}, \quad \mathbf{e}_3 = \mathbf{t}$$

where **n**, **b** and **t** are the unit vectors of the principal normal, the binormal and the tangent respectively. Using the formulae

$$d\mathbf{R}/ds = \mathbf{t}, \quad d\mathbf{t}/ds = k\mathbf{n}, \quad \mathbf{b} = \mathbf{t}\times\mathbf{n}$$
$$ds = g dz, \quad g^2 = (1+x^2), \quad x = \tau r$$

where $k = \tau^2 r/g^2$ is the curvature of the helix, after reduction we obtain an orthogonal matrix from basis $\mathbf{e}_j$ to basis $\mathbf{e}'_i$

$$A = \begin{Vmatrix} -1 & 0 & 0 \\ 0 & -1/g & x/g \\ 0 & x/g & 1/g \end{Vmatrix}$$

We will assume that the material of the cylinder is locally transversely isotropic with axis of symmetry of the mechanical properties directed along the tangent to the helical curve. For the generalized Hooke's law we will use the matrix form of writing the stresses and strains [4]

$$\sigma_i = c_{ij} e_j, \quad c_{ji} = c_{ij}$$

Here

$$\sigma_1 = \sigma_{11}, \quad \sigma_2 = \sigma_{22}, \quad \sigma_3 = \sigma_{33}, \quad \sigma_4 = \sigma_{23}, \quad \sigma_5 = \sigma_{13}, \quad \sigma_6 = \sigma_{12}$$

A similar system of symbols is used for the components of the strain tensor.

For a transversely isotropic material, the elastic properties in basis $\mathbf{e}_j$ are defined by five moduli $c_{11}, c_{12}, c_{13}, c_{33}, c_{44}$, where

$$c_{22} = c_{11}, \quad c_{66} = (c_{11} - c_{12})/2$$
$$c_{15} = c_{16} = c_{25} = c_{26} = c_{25} = c_{26} = c_{35} = c_{36} = 0.$$

We will introduce the following notation

$$\Sigma_1 = \begin{Vmatrix} \sigma_{rr} \\ \sigma_{\theta\theta} \\ \sigma_{zz} \\ \sigma_{\theta z} \end{Vmatrix}, \quad \mathcal{C}'_l = \begin{Vmatrix} c'_{l1} \\ c'_{l2} \\ c'_{l3} \\ 2c'_{l4} \end{Vmatrix}, \quad l = 1, \ldots, 4 \tag{1.2}$$
$$\Sigma_2 = \begin{Vmatrix} \sigma_{rz} \\ \sigma_{r\theta} \end{Vmatrix}, \quad \mathcal{C}'_5 = \begin{Vmatrix} c'_{55} \\ c'_{56} \end{Vmatrix}, \quad \mathcal{C}'_6 = \begin{Vmatrix} c'_{65} \\ c'_{66} \end{Vmatrix}$$

As a result of changing from basis $\mathbf{e}_j$ to basis $\mathbf{e}'_i$ we obtain the following relations of the generalized Hooke's law in the accompanying system of coordinates

$$\Sigma_1 = \mathscr{C}'_1 e_{rr} + \mathscr{C}'_2 e_{\theta\theta} + \mathscr{C}'_3 e_{zz} + \mathscr{C}'_4 e_{\theta z}, \quad \Sigma_2 = \mathscr{C}'_5 e_{rz} + \mathscr{C}'_6 e_{r\theta}$$
$$c'_{11} = c_{11}, \quad c'_{12} = (c_{12} + c_{13}x^2)/g^2$$
$$c'_{13} = (c_{13} + c_{12}x^2)/g^2, \quad c'_{14} = x(c_{13} - c_{12})/g^2$$
$$c'_{22} = [c_{11} + (2c_{13} + 4c_{44})x^2 + c_{33}x^4]/g^4$$
$$c'_{23} = [c_{13} + (c_{11} + c_{33} - 4c_{44})x^2 + c_{13}x^4]/g^4$$
$$c'_{24} = [(c_{13} + 2c_{44} - c_{11})x + (c_{33} - c_{13} - 2c_{44})x^3]/g^4 \tag{1.3}$$
$$c'_{33} = [c_{33} + 2(c_{13} + 2c_{44})x^2 + c_{11}x^4]/g^4$$
$$c'_{34} = [(c_{33} - c_{13} - 2c_{44})x + (c_{44} + 2c_{13} - 2c_{11})x^3]/g^4$$
$$c'_{44} = [c_{44} + (-2c_{13} + c_{11} + c_{33} - 2c_{44})x^2 + c_{44}x^4]/g^4$$
$$c'_{55} = (c_{44} + c_{66}x^2)/g^2, \quad c'_{56} = x(c_{44} - c_{66})/g^2$$
$$c'_{66} = (c_{66} + c_{44}x^2)/g^2$$
$$c'_{ji} = c'_{ij}$$

The components of the strain tensor in the basis of the accompanying system of coordinates can be expressed in terms of the displacements u_r, u_θ, u_z by the following formulae

$$e_{rr} = \partial_r u_r, \quad e_{\theta\theta} = (u_r + \partial_\theta u_\theta)/r, \quad e_{zz} = Du_z$$
$$2e_{r\theta} = \partial_r u_\theta + (\partial_\theta u_r - u_\theta)/r, \quad 2e_{rz} = \partial_r u_z + Du_r, \quad 2e_{z\theta} = \partial_\theta u_z + Du_\theta \tag{1.4}$$

The equilibrium equations in stresses in this case have the form

$$\partial_r(r\sigma_{rr}) - \sigma_{\theta\theta} + \partial_\theta \sigma_{r\theta} + rD\sigma_{rz} = 0$$
$$\partial_r(r\sigma_{r\theta}) + \sigma_{r\theta} + \partial_\theta \sigma_{\theta\theta} + rD\sigma_{\theta z} = 0 \tag{1.5}$$
$$\partial_r(r\sigma_{rz}) + \partial_\theta \sigma_{\theta z} + rD\sigma_{zz} = 0$$

In formulae (1.4) and (1.5)

$$\partial_r = \frac{\partial}{\partial r}, \quad \partial_\theta = \frac{\partial}{\partial \theta}, \quad \partial = \frac{\partial}{\partial z}, \quad D = \partial - \tau\partial_\theta$$

We will assume that the side surface of the cylinder is stress-free, i.e.

$$n_r\sigma_{rr} + n_\theta\sigma_{r\theta} = 0, \quad n_r\sigma_{r\theta} + n_\theta\sigma_{\theta\theta} = 0, \quad n_r\sigma_{rz} + n_\theta\sigma_{z\theta} = 0 \tag{1.6}$$

Introducing the vector $\mathbf{u} = \{u_r, u_\theta, u_z\}$, the problem can be represented in the following vector-operator form

$$M(\partial, \tau)\mathbf{u} \equiv \partial^2 A_0\mathbf{u} + \partial A_1\mathbf{u} + A_2\mathbf{u} = 0 \tag{1.7}$$

$$N(\partial, \tau)\mathbf{u} \equiv (\partial B_0\mathbf{u} + B_1\mathbf{u})|_\Gamma = 0 \tag{1.8}$$

Here (1.7) are the equilibrium equations in displacements, (1.8) are the boundary conditions (1.6), and A_k and B_i are matrix differential operators with respect to the variables r, θ of the zeroth, first and second orders respectively; the specific form of these operators will not be given here due to their complexity and since the method of constructing them is obvious. We note also that, by virtue of relations (1.3), the coefficients of these operators depend on r and τ.

Searching for a solution of problem (1.7), (1.8) in the form

$$\mathbf{u} = \mathbf{a}e^{\gamma z}$$

we obtain the following eigenvalue problem in a section

$$M(\gamma)\mathbf{a} = 0, \quad N(\gamma)\mathbf{a} = 0 \tag{1.9}$$

The common structure of the spectrum, the properties of the system of natural and associated vectors of equations of type (1.9) and the properties of the elementary solutions have been described in sufficient detail in [5, 6]. Hence, we will only describe the group of elementary solutions on which the construction of the solution of the Saint-Venant problem of the stretching, twisting and bending of a cylinder with helical anisotropy is based. We will call this group of elementary solutions the Saint-Venant elementary solutions.

2. THE SAINT-VENANT SOLUTION

We will initially construct the Saint-Venant elementary solutions. Retaining the basic idea [3], we will write the vector of the solid displacement in the accompanying system of coordinates. We have

$$\begin{aligned}
\overset{0}{u}_r &= C_1e^{i\psi} + C_2e^{-i\psi} + zC_3e^{i\psi} + zC_4e^{-i\psi}\\
\overset{0}{u}_\theta &= iC_1e^{i\psi} - iC_2e^{-i\psi} + izC_3e^{i\psi} - izC_4e^{-i\psi} - C_6r\\
\overset{0}{u}_z &= -C_3re^{i\psi} - C_4re^{-i\psi} + C_5\\
\psi &= \tau z + \theta\\
C_1 &= \frac{1}{2}(\overset{0}{a}_1 - i\overset{0}{a}_2), \quad C_2 = \bar{C}_1, \quad C_3 = \frac{1}{2}(\omega_2 + i\omega_1), \quad C_4 = \bar{C}_3\\
C_5 &= \overset{0}{a}_3, \quad C_6 = \omega_3
\end{aligned} \tag{2.1}$$

Here $\overset{0}{a}_k$ and ω_k are the projections of the vector of the translational displacement and rotation of the cylinder as a rigid body onto the axis of the fundamental system of coordinates.

It follows from formulae (2.1) that

$$\gamma_0 = 0, \quad \gamma_1 = i\tau, \quad \gamma_{-1} = -i\tau$$

Natural values of eigenvalue problem (1.9) exist. The structure of the root subspaces of eigenvalues remains the same as in the case of a naturally twisted rod [3], which enables us, in the case of a cylinder with a circular cross-section $r_1 \leqslant r \leqslant r_2$, to write the elementary Saint-Venant solutions in the form

$$\begin{aligned}
&\mathbf{u}_1(z) = \mathbf{a}_5, \quad \mathbf{u}_2(z) = \mathbf{a}_6, \quad \mathbf{u}_3(z) = e^{i\psi}\mathbf{a}_3\\
&\mathbf{u}_4(z) = \bar{\mathbf{u}}_3(z), \quad \mathbf{u}_5(z) = e^{i\psi}(z\mathbf{a}_3 + \mathbf{a}_5), \quad \mathbf{u}_6(z) = \bar{\mathbf{u}}_5(z)\\
&\mathbf{u}_7(z) = z\mathbf{a}_1 + \mathbf{a}_7, \quad \mathbf{u}_8(z) = z\mathbf{a}_2 + \mathbf{a}_8\\
&\mathbf{u}_9(z) = e^{i\psi}\left(\frac{z^2}{2}\mathbf{a}_3 + z\mathbf{a}_5 + \mathbf{a}_9\right), \quad \mathbf{u}_{10}(z) = \bar{\mathbf{u}}_9(z)\\
&\mathbf{u}_{11}(z) = e^{i\psi}\left(\frac{z^3}{6}\mathbf{a}_3 + \frac{z^2}{2}\mathbf{a}_5 + z\mathbf{a}_9 + \mathbf{a}_{11}\right), \quad \mathbf{u}_{12}(z) = \bar{\mathbf{u}}_{11}(z)
\end{aligned} \tag{2.2}$$

Here

$$\begin{aligned}
&\mathbf{a}_1 = \{0, 0, 1\}, \quad \mathbf{a}_2 = \{0, r, 0\}, \quad \mathbf{a}_3 = \{1, i, 0\}, \quad \mathbf{a}_4 = \bar{\mathbf{a}}_3\\
&\mathbf{a}_5 = \mathbf{a}_6 = \{0, 0, -r\}, \quad \mathbf{a}_l = \{a_{r,l}, a_{\theta,l}, a_{z,l}\}, \quad l = 7, 8\\
&\mathbf{a}_9 = \{a_{r,9}, ia_{\theta,9}, a_{z,9}\}, \quad \mathbf{a}_{10} = \bar{\mathbf{a}}_9, \quad \mathbf{a}_{11} = \{ia_{r,7}, a_{\theta,7}, ia_{z,7}\}, \quad \mathbf{a}_{12} = \bar{\mathbf{a}}_{11}
\end{aligned} \tag{2.3}$$

In the case of a cylinder with a circular cross-section considered, the components of the vectors $\mathbf{a}_l$ ($l = 7, \ldots, 12$) depend only on r. To formulate the boundary-value problems for determining them we must return to relations (1.2)–(1.6).

We will consider the problem of determining the components of the vector $\mathbf{a}_7$ in more detail. Using relations (1.2) ad (1.4), we obtain the following expressions for the components of the stress tensor

$$\Sigma_{1,7} = \mathscr{C}_1'\frac{da_{r,7}}{dr} + \frac{1}{r}\mathscr{C}_2'a_{r,7} + \mathscr{C}_3', \quad \Sigma_{2,7} = \mathscr{C}_5'\frac{da_{z,7}}{dr} + \mathscr{C}_6'\left(\frac{da_{\theta,7}}{dr} - \frac{a_{\theta,7}}{r}\right) \tag{2.4}$$

From the equilibrium equations (1.5) and boundary conditions (1.6) ($n_r = 1, n_0 = 0$) we have

$$\partial_r(r\sigma_{rr,7}) - \sigma_{\theta\theta,7} = 0, \quad \sigma_{rr,7}(r_\alpha) = 0 \tag{2.5}$$

$$\partial_r(r\sigma_{r\theta,7}) + \sigma_{r\theta,7} = 0, \quad \sigma_{r\theta,7}(r_\alpha) = 0 \tag{2.6}$$

$$\partial_r(r\sigma_{rz}) = 0, \quad \sigma_{rz}(r_\alpha) = 0 \tag{2.7}$$

It follows from Eqs (2.6) and (2.7) that

$$\sigma_{r\theta,7} = \sigma_{rz,7} = 0$$

It follows from these relations and expressions (2.6) for $\sigma_{r\theta,7}$, $\sigma_{rz,7}$ that

$$a_\theta = X_1 r + X_0, \quad a_z = X_2$$

where X_0, X_1 and X_2 are arbitrary constants, which can be equated to zero.

Substituting the expressions for $\sigma_{rr,7}$, $\sigma_{\theta\theta,7}$ from (2.4) and (2.5), to determine $a_{r,7}$ we obtain a boundary-value problem for the second-order ordinary differential operator

$$Za_{r,7} = F_7, \quad la_{r,7}|_{r=r_\alpha} = f_{\alpha,7}$$

$$Za \equiv \frac{d}{dr}\left(rc_{11}'\frac{da}{dr} + c_{12}'a\right) - c_{12}'\frac{da}{dr} - \frac{1}{r}c_{22}'a, \quad la \equiv c_{11}'\frac{da}{dr} + \frac{1}{r}c_{12}'a \tag{2.8}$$

$$F_7 = -\frac{d(rc_{13}')}{dr} + c_{23}', \quad f_{\alpha,7} = -c_{13}'(r_\alpha)$$

In a similar way we obtain that

$$\mathbf{a}_8 = \{a_{r,8}, 0, 0\}$$

In this case also the determination of $a_{r,8}$ reduces to boundary-value problem (2.8) with the new right-hand sides

$$F_8 = -\frac{d(r^2c_{14}')}{dr} + rc_{24}', \quad f_{\alpha,8} = -r_\alpha c_{14}'(r_\alpha)$$

The stresses are given by the following formulae

$$\Sigma_{1,8} = \mathscr{C}_1'\frac{da_{r,8}}{dr} + \frac{1}{r}\mathscr{C}_2'a_{r,8} + r\mathscr{C}_4', \quad \Sigma_{2,8} = 0 \tag{2.9}$$

We will introduce the following notation

$$\mathscr{B}_{1,l} = \begin{Vmatrix} b_{rr,l} \\ b_{\theta\theta,l} \\ b_{zz,l} \\ b_{\theta z,l} \end{Vmatrix}, \quad \mathscr{B}_{2,l} = \begin{Vmatrix} b_{rz,l} \\ b_{r\theta,l} \end{Vmatrix}, \quad l = 9, 11$$

For the elementary solution $\mathbf{a}_9$, the stresses corresponding to it are given by the following formulae

$$\Sigma_{1,9} = e^{i\Psi}\mathcal{B}_{1,9}, \quad \Sigma_{2,9} = e^{i\Psi}\mathcal{B}_{2,9} \tag{2.10}$$

$$\begin{aligned} B_{1,9} &= \mathcal{C}'_1\frac{da_{r,9}}{dr} + \mathcal{C}'_2\frac{a_{r,9}-a_{\theta,9}}{r} - \mathcal{C}'_3 r - \mathcal{C}'_4\frac{a_{z,9}}{r} \\ B_{2,9} &= \mathcal{C}'_5\frac{da_{z,9}}{dr} + \mathcal{C}'_6\left(\frac{da_{\theta,9}}{dr} + \frac{a_{r,9}-a_{\theta,9}}{r}\right) \end{aligned} \tag{2.11}$$

In this case, all the components of the vector $\mathbf{a}_9$ are non-zero and are determined by integrating the boundary-value problem obtained after substituting expressions (2.11) into the following relations

$$\begin{aligned} &\frac{d(rb_{rr,9})}{dr} - b_{r\theta,9} - b_{\theta\theta,9} = 0, \quad b_{rr,9}(r_\alpha) = 0 \\ &\frac{d(rb_{r\theta,9})}{dr} + b_{r\theta,9} + b_{\theta\theta,9} = 0, \quad b_{r\theta,9}(r_\alpha) = 0 \\ &\frac{d(rb_{rz,9})}{dr} + b_{z\theta,9} = 0, \quad b_{rz,9}(r_\alpha) = 0 \end{aligned} \tag{2.12}$$

As follows from the first two equations of (2.12), this system has a first integral

$$b_{rr,9} + b_{r\theta,9} = 0$$

For the elementary solution $\mathbf{a}_{11}$, the stresses corresponding to it are given by the formulae

$$\Sigma_{1,11} = e^{i\Psi}(z\mathcal{B}_{1,9} + i\mathcal{B}_{1,11}), \quad \Sigma_{2,11} = e^{i\Psi}(iz\mathcal{B}_{2,9} + \mathcal{B}_{2,11}) \tag{2.13}$$

$$\begin{aligned} B_{1,11} &= \mathcal{C}'_1\frac{da_{r,11}}{dr} + \mathcal{C}'_2\frac{a_{r,11}-a_{\theta,11}}{r} + \mathcal{C}'_4\frac{a_{z,11}}{r} + \mathcal{C}'_3 a_{z,9} + \mathcal{C}'_4 a_{\theta,9} \\ B_{2,11} &= \mathcal{C}'_5\frac{da_{z,11}}{dr} + \mathcal{C}'_6\left(\frac{da_{\theta,11}}{dr} + \frac{a_{r,11}+a_{\theta,11}}{r}\right) + \mathcal{C}'_5 a_{r,9} \end{aligned} \tag{2.14}$$

Substituting expressions (2.13) into the equilibrium equations and boundary conditions, we obtain the relations

$$\begin{aligned} &\frac{d(rb_{rr,11})}{dr} + b_{r\theta,11} - b_{\theta\theta,11} + rb_{rz,9} = 0, \quad b_{rr,11}(r_\alpha) = 0 \\ &\frac{d(rb_{r\theta,11})}{dr} + b_{r\theta,11} - b_{\theta\theta,11} + rb_{z\theta,9} = 0, \quad b_{r\theta,11}(r_\alpha) = 0 \\ &\frac{d(rb_{rz,11})}{dr} - b_{z\theta,11} + rb_{zz,9} = 0, \quad b_{rz,11}(r_\alpha) = 0 \end{aligned} \tag{2.15}$$

These relations, after substituting expressions (2.14) into them, lead to a boundary-value problem for three second-order differential equations in the components $a_{r,11}, a_{\theta,11}, a_{z,11}$. The following first integral is obtained from the first two equations of (2.15)

$$b_{rr,11} - b_{r\theta,11} = \frac{1}{r}\int_{r_1}^{r}(b_{rz,9} - b_{\theta z,9})r\,dr$$

We will now consider the Saint-Venant problem. We will assume that the following boundary conditions are specified on the ends of the cylinder

$$u_j = 0 \quad \text{when} \quad z = 0 \tag{2.16}$$

$$\sigma_{j3} = p_j \quad \text{when} \quad z = L \tag{2.17}$$

Here u_j and p_j are the projections of the displacement vector and of the vector of the specified external forces onto the axis of the fundamental system of coordinates.

The Saint-Venant solution will be sought in the form

$$\mathbf{u} = \sum_{l=1}^{6} C_l \mathbf{u}_l(z) + \sum_{l=7}^{12} C_l \mathbf{u}_l(z-L) \tag{2.18}$$

$$\boldsymbol{\sigma} = \{\sigma_{rz}, \sigma_{r\theta}, \sigma_{zz}\} = \sum_{l=7}^{12} C_l \boldsymbol{\sigma}_l(z-L) \tag{2.19}$$

By satisfying boundary condition (2.19) in the integral sense and using the relations of generalized orthogonality [6], we obtain

$$\begin{aligned} &d_{11}C_7 + d_{12}C_8 = Q_3, \quad d_{12}C_7 + d_{22}C_8 = M_3 \\ &d_{33}C_9 + d_{35}C_{11} = M_2 + iM_1, \quad d_{33}C_{11} = -Q_1 + iQ_2 \\ &C_8 = \bar{C}_7, \quad C_{10} = \bar{C}_9 \end{aligned} \tag{2.20}$$

Here

$$\begin{aligned} &d_{11} = 2\pi\int_{r_1}^{r_2} r\sigma_{zz,7}dr, \quad d_{12} = 2\pi\int_{r_1}^{r_2} r\sigma_{zz,8}dr, \quad d_{22} = 2\pi\int_{r_1}^{r_2} r^2\sigma_{rz,8}dr \\ &d_{33} = -2\pi\int_{r_1}^{r_2} r^2 b_{zz,9}dr, \quad d_{35} = 2\pi i\int_{r_1}^{r_2} r^2 b_{zz,11}dr \end{aligned} \tag{2.21}$$

and Q_j and M_j are the projections of the principal vector and the principal moment of the external forces onto the axis of the fundamental system of coordinates.

Relations (2.20) and (2.21) enable us to give the following interpretation to the elements of the stiffness matrix: d_{11} is the extension–compression stiffness of the rod, d_{22} is the twisting stiffness, d_{12} is the extension–compression and twisting coupling coefficient and d_{33} is the bending stiffness.

Relations (2.20) and (2.21) are exact in the sense that the boundary-layer part of the solution of the three-dimensional problem, which is produced by the self-balancing part of the external stresses (2.17), has no effect on their form.

The constants $C_1, \ldots, C_6$ are determined when boundary conditions (2.16) are satisfied. Their exact values can only be determined by solving infinite systems of algebraic equations, similar to those constructed previously in [7] for an isotropic cylinder. In the case of long cylinders ($h = r_2/L \ll 1$), these constants are defined by the following approximate formulae

$$\begin{aligned} &C_1 = -LC_7, \quad C_2 = -LC_8, \quad C_3 = -\frac{1}{2}L^2C_{11}e^{-i\tau L}, \quad C_4 = \bar{C}_3 \\ &C_5 = \left(\frac{1}{6}L^3C_{11} - \frac{1}{2}L^2C_9\right)e^{-i\tau L}, \quad C_6 = \bar{C}_5 \end{aligned} \tag{2.22}$$

3. METHODS OF CONSTRUCTING ELEMENTARY SAINT-VENANT SOLUTIONS AND THE RESULTS OF A NUMERICAL ANALYSIS

We will now consider the problem of constructing solutions of problems (2.8), (2.11), (2.12) and (2.15), and using them to calculate the elements of the stiffness matrix d_{ij}.

We will first consider the construction of analytical solutions by the small-parameter method, assuming that the dimensionless parameter $\tau_0 = \tau r_2 \ll 1$. These solutions, in addition to the fact that they enable us to obtain a clear representation of the effect of the different parameters of the problem on its solution, are useful as a test for the numerical solution.

To construct approximate solutions we use formulae (1.3) and expand c'_{ml} in series in τ. Retaining the principal terms of the expansions, we obtain

$$\begin{aligned} c'_{11} &= c'_{22} = c_{11}, \quad c'_{12} = c_{12}, \quad c'_{13} = c'_{23} = c_{13} \\ c'_{33} &= c_{33}, \quad c'_{44} = c'_{55} = c_{44}, \quad c'_{66} = c_{66} \\ c'_{14} &= \tau r(c_{13} - c_{12}), \quad c'_{24} = \tau r(c_{13} + 2c_{44} - c_{11}) \\ c'_{34} &= \tau r(c_{33} - c_{13} - 2c_{44}), \quad c'_{56} = \tau r(c_{44} - c_{66}) \end{aligned} \tag{3.1}$$

We will seek a solution of boundary-value problem (2.8) in the form

$$a_{r,7} = a_7^{(0)} + \tau a_7^{(1)} + \dots \tag{3.2}$$

After substituting expressions (3.1) and (3.2) into relations (2.8), we obtain the following boundary-value problem for determining the principal term

$$\frac{d^2 a_7^{(0)}}{dr^2} + \frac{1}{r}\frac{da_7^{(0)}}{dr} - \frac{a_7^{(0)}}{r^2} = 0 \quad \left(c_{11}\frac{da_7^{(0)}}{dr} + c_{12}\frac{a_7^{(0)}}{r}\right)_{r = r_\alpha} = -c_{13}$$

We will present the final form of the solution, omitting the further elementary stages of the integration. We have

$$a_{r,7} = -\nu' r + O(\tau^2) \tag{3.3}$$

In a similar way we obtain

$$\begin{aligned} a_{r,8} &= \frac{\tau r_2}{8c_{11}}\left\{K_0\frac{r^3}{r_2^2} + K_1\left[\frac{(1+\rho^2)r}{c_{11}+c_{12}} + \frac{\rho^2 r_2^2}{r(c_{11}-c_{12})}\right]\right\} + O(\tau^3) \\ a_{r,9} &= \frac{\nu' r^2}{2} + O(\tau^2), \quad a_{\theta,9} = -\frac{\nu' r^2}{2} + O(\tau^2) \\ a_{z,9} &= \frac{\tau K_2}{8}\left[r^3 - 3(r_1^2 + r_2^2)r - \frac{3r_1^2 r_2^2}{r}\right] + O(\tau^3) \\ a_{z,11} &= \frac{\tau K_2}{8}\left[r^3 - \kappa(r_1^2 + r_2^2)r - \frac{\kappa r_1^2 r_2^2}{r}\right] + O(\tau^2) \\ a_{r,11} &= O(\tau), \quad a_{\theta,11} = O(\tau) \end{aligned} \tag{3.4}$$

In formulae (3.3) and (3.4)

$$\begin{aligned} E' &= c_{33} - 2\nu' c_{13}, \quad G' = c_{44} \\ K_0 &= 3c_{12} - 2c_{13} - c_{11} + 2c_{44} \\ K_1 &= -3c_{12}^2 + 3c_{11}^2 - 2c_{13}(c_{11} - c_{12}) - 2c_{44}(2c_{12} + 6c_{11}) \\ K_2 &= E' - 2(1+\nu')G', \quad \kappa = 3K_2 + \nu' G'/2, \quad \rho = r_1/r_2 \\ \nu' &= (c_{11} + c_{12})/c_{13} \end{aligned}$$

We will present expressions for the principal terms of the components of the stress tensors. For $l = 7, 8$, we have

$$\sigma_{zz,7} = E' + O(\tau^2), \quad \sigma_{\theta z,8} = E' r + O(\tau^2), \quad \sigma_{r\theta,l} = \sigma_{rz,l} \equiv 0$$

$$\sigma_{rr,7} \simeq \sigma_{\theta\theta,7} \simeq O(\tau^2), \quad \sigma_{\theta z,7} \simeq \sigma_{rr,8} \simeq \sigma_{\theta\theta,8} \simeq \sigma_{zz,8} \simeq O(\tau)$$

For $l = 9, 11$, we will confine ourselves to giving the principal terms of the expansions of the traces of the stresses

$$b_{zz,9} = -E'r + O(\tau^2)$$

$$b_{rz,9} = \frac{\tau K_2}{8}\left(r^2 - r_1^2 - r_2^2 + \frac{r_1^2 r_2^2}{r^2}\right) + O(\tau^3)$$

$$b_{\theta z,9} = -\frac{\tau K_2}{8}\left(3r^2 - r_1^2 - r_2^2 - \frac{r_1^2 r_2^2}{r^2}\right) + O(\tau^3)$$

$$b_{rr,9} \simeq b_{\theta\theta,9} \simeq b_{r\theta,9} \simeq O(\tau^2)$$

$$b_{rz,11} = G'\left[\left(\frac{3K_2}{8} + \frac{\nu'}{2}\right)r^2 - \kappa(r_1^2 + r_2^2)r + \frac{\kappa r_1^2 r_2^2}{r^2}\right]$$

$$b_{\theta z,11} = G'\left[\left(\frac{K_2}{8} - \frac{\nu'}{2}\right)r^2 - \kappa(r_1^2 + r_2^2)r - \frac{\kappa r_1^2 r_2^2}{r^2}\right]$$

$$b_{rr,11} \simeq b_{\theta\theta,11} \simeq b_{zz,11} \simeq b_{r\theta,11} \simeq O(\tau)$$

The principal terms of the expansions of d_{ij} have the form

$$d_{11}^0 = SE', \quad d_{22}^0 = c_{44}J_p$$

$$d_{12}^0 = \tau J_p K_2, \quad d_{33} = E'J_p, \quad d_{31} = O(\tau)$$

$$S = \pi(r_2^2 - r_1^2), \quad J_p = \pi(r_2^4 - r_1^4)/2$$

The formula for d_{12}^0 gives a representation of the effect of the parameters of the problem on the interaction between twisting and tension-compression for small values of τ_0. In particular, the coefficient K_2 represents the degree of anisotropy of a transversely isotropic material. In this formula E', G' and ν' are the technical constants of elasticity [4]: the tensile and shear moduli and Poisson's ratio respectively. Note that $K_2 = 0$ for an isotropic material.

To estimate the effect of the parameters τ_0 and ρ on the elements d_{ij} over a wide range of variation of these parameters, we carried out a series of calculations based on numerical integration of the problems of determining the components $a_{r,7}$ and $a_{r,8}$. We chose for the calculations a composite fibre material will the following characteristics of elasticity

$E' = 3.685 \times 10^9$ Pa, $E = 9.398 \times 10^9$ Pa, $G' = 1.272 \times 10^9$ Pa, $\nu' = 0.0147$ and $\nu = 0.375$

For calculations on the constants c_{ij} we obtain

$c_{11} = 5.294 \times 10^9$ Pa, $c_{12} = 1.770 \times 10^9$ Pa, $c_{13} = 0.104 \times 10^9$ Pa, $c_{33} = 3.688 \times 10^9$ Pa and $c_{44} = G'$

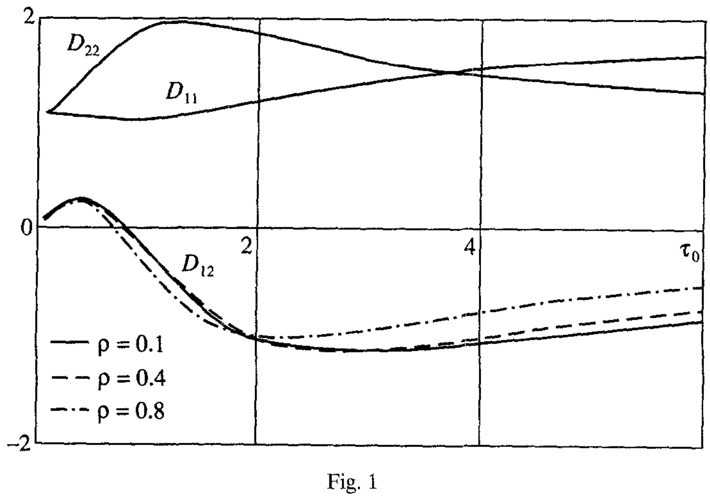

Fig. 1

Figure 1 shows graphs of the dimensionless elements of the stiffness matrix

$$D_{11} = \frac{d_{11}}{d_{11}^0}, \quad D_{22} = \frac{d_{22}}{d_{22}^0}, \quad D_{12} = \frac{J_{12}}{J_p B}$$

as a function of τ_0 for $\rho = 0.1$ (the continuous curves); we also show graphs of D_{12} against τ_0 for $\rho = 0.4$ and 0.8 (the dashed curve and the dash–dot curve).

The curves describing the behaviour of D_{12} indicate that when a cylinder is stretched, an "untwisting" effect occurs for small values of the parameter, while a "twisting" effect occurs for fairly large values of the parameter.

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